

Haya Alshamsi

Exam I: MTH 111, Fall 2017

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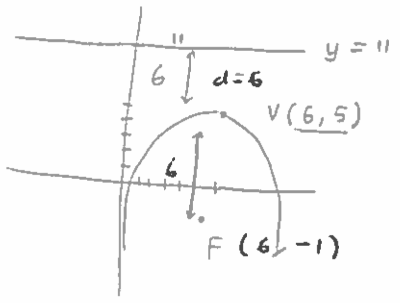
Points = $\frac{70}{70}$ Excellent

QUESTION 1. (6 points) Given $y = 11$ is the directrix of of a parabola that has the point $(6, 5)$ as its vertex point.
a) Find the equation of the parabola $\boxed{d=6}$

$$-4d(y-y_1) = (x-x_1)^2$$

$$-4(6)(y-5) = (x-6)^2$$

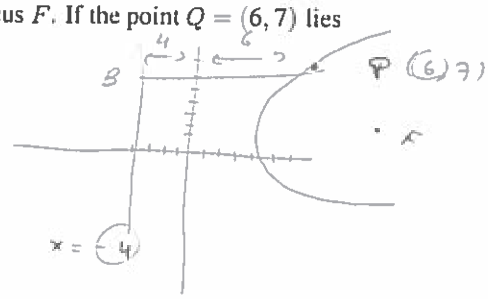
$$-24(y-5) = (x-6)^2$$



b) Find the focus of the parabola.

$F\ (6,-1)$

QUESTION 2. (3 points) Given that $x = -4$ is the directrix of a parabola that has focus F. If the point $Q = (6, 7)$ lies on the curve of the parabola, find $|QF|$ (i.e., find the distance between F and Q).



$|QL| = |QF|$

$|QB| = |QF|$

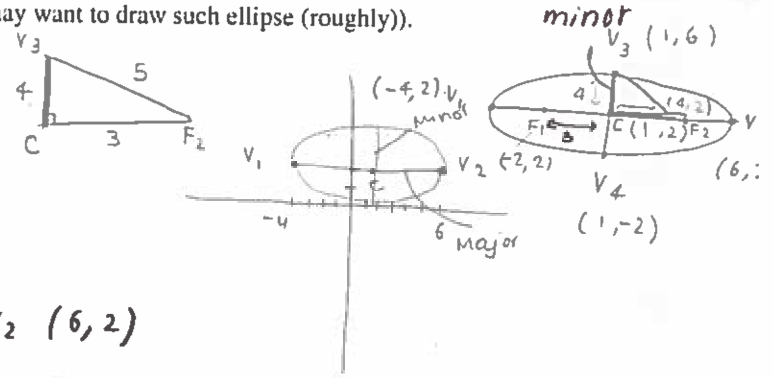
$|QF| = 10$ units

QUESTION 3. (8 points) Given $(-4, 2)$, $(6, 2)$ are the vertices of the major axis of an ellipse (recall major axis is the longer axis) and $(4, 2)$ is one of the foci.

(i) Find the vertices of the minor axis (shorter axis). (you may want to draw such ellipse (roughly)).

$V_3\ (1,6)$

$V_4\ (1,-2)$



(ii) Find the ellipse-constant K. $C\ (1,2)$; $V_2\ (6,2)$

$\frac{K}{2} = 5 \Rightarrow \boxed{K=10}$

(iii) Find the second foci of the ellipse.

$F_1\ (-2,2)$

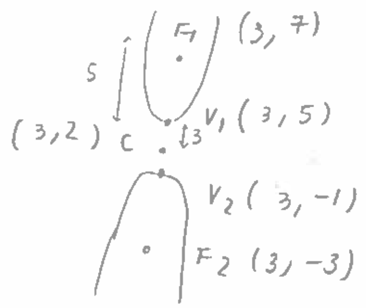
(iv) Find the equation of the ellipse. horizontal ellipse ; $K=10$; ~~$(b=3)$~~ $b=4$

$$\frac{(x-1)^2}{25} + \frac{(y-2)^2}{16} = 1$$

QUESTION 4. (8 points)

Draw roughly the hyperbola $\frac{(y-2)^2}{9} - \frac{(x-3)^2}{16} = 1$. Then find

positive $y \Rightarrow$



a) The hyperbola-constant K.

$\left(\frac{K}{2}\right)^2 = 9 \rightarrow \frac{K}{2} = 3$

$\boxed{K = 6}$

b) The two vertices of the hyperbola.

$V_1 \ (3,5)$

$V_2 \ (3,-1)$

c) The foci of the hyperbola. $|CF_1| = \sqrt{9+16} = 5$

$F_1 \ (3,7)$

$F_2 \ (3,-3)$

QUESTION 5. (3 points) Given $y = 2x^2 + 6x + 8$ is an equation of a parabola. Write the equation of the parabola in the standard form

$y = 2x^2 + 6x + 8$

$y = 2(x^2 + 3x + 4)$

$y = 2(x + 3x + \frac{9}{4} - \frac{9}{4} + 4)$

$y = 2(x + \frac{3}{2})^2 + \frac{7}{2}$

$y - \frac{7}{2} = 2(x + \frac{3}{2})^2$

$\frac{1}{2}(y - \frac{7}{2}) = (x + \frac{3}{2})^2$

QUESTION 6. (6 points) Given $(x+2) = \frac{1}{12}(y-3)^2$ is an equation of a parabola

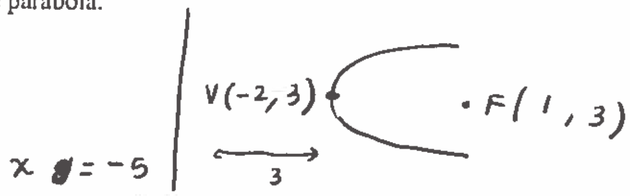
a) Draw roughly the parabola.

$12(x+2) = (y-3)^2$

+ (positive x) $\Rightarrow$

$4d = 12$

$d = \frac{12}{4} = 3$



b) Find the equation of the directrix line.

$\boxed{x = -5}$

c) Find the focus of the parabola.

F (1,3)

QUESTION 7. (10 points) a) Given two lines $L_1 : x = 2t, y = 2t + 3, z = -t + 1$ and $L_2 : \boxed{x = 6w - 6}$ and $y = 6w - 3, z = -3w + 4$. Is L_1 parallel to L_2? EXPLAIN clearly. If not, can you say something about L_1 and L_2?

$L_1 : \begin{cases} x = 2t \\ y = 2t + 3 \\ z = -t + 1 \end{cases} \; ; t \in \mathbb{R}$

$L_2 : \begin{cases} x = 6w - 6 \\ y = 6w - 3 \\ z = -3w + 4 \end{cases} \; ; w \in \mathbb{R}$

$D_1 : \langle 2, 2, -1 \rangle$ ($\times 3$, $\times 3$, $\times 3$)

$D_2 : \langle 6, 6, -3 \rangle$

$D_1 \parallel D_2$

→ choose a point on L_1 ;

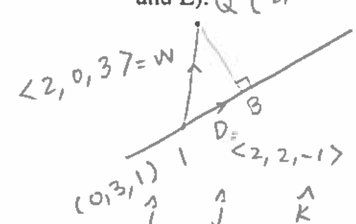
$t = 0 \rightarrow (0, 3, 1)$

→ now check if it's on L_2 ;

$0 = 6w - 6$, $\boxed{w = 1}$ | $3 = 6w - 3$, $\boxed{w = 1}$ | $1 = -3w + 4$, $\boxed{w = 1}$

It is on L_2 ⇒ The lines are not parallel. They are collinear/on top of each other

b) Let L be the line L_1 as in (a). Given that the point $Q = (2, 3, 4)$ does not lie on L. Find $|QL|$ (distance between Q and L). $Q\,(2,3,4)$



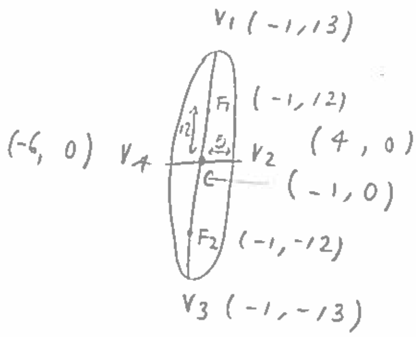
$|QL| = |QB| = \dfrac{|D \times W|}{|D|}$

$|QB| = \dfrac{\sqrt{36 + 64 + 16}}{\sqrt{4 + 4 + 1}} = \dfrac{2\sqrt{29}}{3}$ units

$D \times W = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & -1 \\ 2 & 0 & 3 \end{vmatrix} = 6\hat{i} - 8\hat{j} - 4\hat{k}$

QUESTION 8. (6 points) Consider the ellipse $\dfrac{(x+1)^2}{25} + \dfrac{y^2}{169} = 1.$

a) Draw roughly such ellipse



$\left(\frac{k}{2}\right)^2 = 169$

$\frac{k}{2} = 13$

$\boxed{k = 26}$

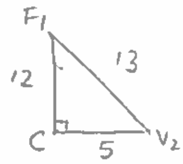
$b^2 = 25$

$\boxed{b = 5}$

b) Find the foci of the ellipse.

$F_1 \; (-1, 12)$

$F_2 \; (-1, -12)$



c) Find all 4 vertices of of the ellipse

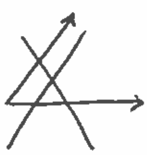
$V_1 \; (-1, 13)$

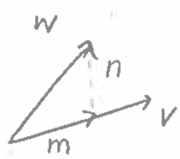
$V_3 \; (-1, -13)$

$V_2 \; (4, 0)$

$V_4 \; (-6, 0)$

QUESTION 9. (8 points) Let $V = \langle 1, -2, 2 \rangle, w = \langle 2, -2, 0 \rangle$ (where V and W have the same initial point). Write W as a sum of two vectors m and n, where m is parallel to V and n is perpendicular to v.





$W = m + n$

$W = \text{proj}_V W + n$

$n = W - \text{proj}_V W$

$n = \langle 2, -2, 0 \rangle - \dfrac{W \cdot V}{|V|^2} V$

$n = \langle 2, -2, 0 \rangle - \dfrac{2 + 4}{9} \langle 1, -2, 2 \rangle$

$n = \langle 2, -2, 0 \rangle - \dfrac{6}{9} \langle 1, -2, 2 \rangle$

$n = \langle 2, -2, 0 \rangle - \frac{2}{3} \langle 1, -2, 2 \rangle$

$n = \langle 2, -2, 0 \rangle - \langle \frac{2}{3}, -\frac{4}{3}, \frac{4}{3} \rangle$

$\boxed{n = \langle \frac{4}{3}, -\frac{2}{3}, -\frac{4}{3} \rangle}$

$m = \text{proj}_V W = \frac{2}{3} \langle 1, -2, 2 \rangle$

$\boxed{m = \langle \frac{2}{3}, \frac{-4}{3}, \frac{4}{3} \rangle}$

QUESTION 10. (12 points)

a) Convince me that $q_1 = (0, 4, 2)$, $q_2 = (2, 1, -1)$, and $q_3 = (2, 3, 5)$ are not co-linear

$\overrightarrow{Q_1Q_2} = \langle 2, -3, -3 \rangle$

$\overrightarrow{Q_1Q_3} = \langle 2, -1, 3 \rangle$

$$\overrightarrow{Q_1Q_2} \times \overrightarrow{Q_1Q_3} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -3 \\ 2 & -1 & 3 \end{vmatrix}$$

$= -12\hat{i} - 12\hat{j} + 4\hat{k}$

The cross-product is not a zero-vector $\Rightarrow$ The points are not collinear

b) Find the area of the triangle with vertices q_1, q_2, q_3. (q_1, q_2, q_3 as in (a))

$A_\Delta = \frac{1}{2} |\overrightarrow{Q_1Q_2} \times \overrightarrow{Q_1Q_3}|$

$A_\Delta = \frac{1}{2} \sqrt{144 + 144 + 16} = \frac{1}{2}(4\sqrt{19}) = 2\sqrt{19}$ units^2

c) Find a vector F that is perpendicular to both vectors $\overrightarrow{q_1q_2}$ and $\overrightarrow{q_1q_3}$. (q_1, q_2, q_3 as in (a))

$F = |\overrightarrow{Q_1Q_2} \times \overrightarrow{Q_1Q_3}| = -12\hat{i} - 12\hat{j} + 4\hat{k} = \langle -12, -12, 4 \rangle.$

d) Convince me that the line $L_1 : x = 2t + 1, y = -t + 3, z = 4t + 1$ ($t \in R$) is perpendicular to the line $L_2 : x = -2w + 5, y = 4w - 5, z = 2w - 3$ ($w \in R$).

$L_1 : \begin{cases} x = 2t + 1 \\ y = -t + 3 \\ z = 4t + 1 \end{cases} ; t \in \mathbb{R}$ $\qquad D_1 : \langle 2, -1, 4 \rangle$

$L_2 : \begin{cases} x = -2w + 5 \\ y = 4w - 5 \\ z = 2w - 3 \end{cases} ; w \in \mathbb{R}$ $\qquad D_2 : \langle -2, 4, 2 \rangle$

$D_1 \cdot D_2 = 2(-2) - 4 + 8 = -4 - 4 + 8 = -8 + 8 = 0.$

$\rightarrow$ check if they intersect:

$\begin{cases} 2t + 1 = -2w + 5 \\ -t + 3 = 4w - 5 \end{cases} \rightarrow \begin{cases} 2t + 2w = 4 \\ -t - 4w = -8 \end{cases} \rightarrow \begin{cases} t + w = 2 \\ -t - 4w = -8 \end{cases} \rightarrow \boxed{w = 2 - t}$

$-t - 4(2 - t) = -8$

$-t - 8 + 4t = -8$

$3t - 8 = -8$

$3t = 0$

$\boxed{t = 0}$

$\boxed{w = 2}$

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The two lines intersect at $(1, 3, 1)$ $\Rightarrow$ The two lines are perpendicular.

$z = 4(0) + 1$ $\qquad$ $\boxed{z = 1}$

$z = 2(2) - 3$ $\qquad$ $\boxed{z = 4 - 3 = 1}$